A structure theorem for rationalizability in the normal form of dynamic games*

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Abstract

We prove the structure theorem for rationalizability due to Weinstein and Yildiz (2007) in any finite extensive-form game with perfect recall and suitably rich payoffs. We demonstrate that the ties induced by the extensive form do not change the result of Weinstein and Yildiz (2007). Specifically, like Weinstein and Yildiz (2007), we adopt the normal form concept of interim correlated rationalizability and we assume that players have no relevant knowledge of the extensive-form payoff structure. The extensive-form result is weaker in the sense that while the result of Weinstein and Yildiz (2007) does not depend on the second assumption, our result does. Our result implies that without restrictions on players’ knowledge of payoffs, the dynamic structure of extensive-form games offers no force for robust refinements of rationalizability. We also strengthen the main selection result of Weinstein and Yildiz (2007) along a different direction (and obtain a similarly strengthened result in our context). They show that for any rationalizable action $a$ of any finite type $t$, there is a sequence of types with a unique rationalizable action equal to $a$ that converges to $t$. We show this holds for all types (not only those that are finite).

Keywords: Rationalizability, Incomplete information, Robustness, Universal type space, Higher order beliefs, Extensive-form games

JEL classification: C72, D80
1 Introduction

While rationalizability is one of the major solution concepts in game theory, economic models typically have a large number of rationalizable outcomes. Weinstein and Yildiz (2007) (hereafter, WY) establish a striking structure on the correspondence from (Harsanyi) types to sets of (interim correlated) rationalizable (ICR) outcomes: for any rationalizable action $a$ of any finite type, we can perturb (in product topology on the universal type space) the beliefs of the type in such a way that $a$ is uniquely rationalizable for the new type. By upper hemicontinuity of rationalizability (Dekel et al. (2006)), this unique outcome will be robust to further small changes. Hence, types with unique rationalizable actions are generic (i.e. open and dense) in the universal type space and no refinements of rationalizability are robust under this notion of perturbation.

Their results rely on a "richness" assumption on the game, namely that every action for every player is strictly dominant for some payoff parameter. As WY observe, this assumption holds — in simultaneous-move games — if we assume away any knowledge on payoffs. However, it essentially rules out all dynamic games. Since an important implication of the structure theorem of WY is that no refinement can produce robust predictions beyond rationalizability, it is natural to ask whether there is some weaker condition which ensures their result and is suitable for dynamic games.\footnote{WY mention that in a dynamic game, one can introduce trembles and use a reduced form to satisfy their richness assumption. However, introducing trembles changes the game, and hence makes it unclear whether some refinements may still produce robust predictions beyond rationalizability in the game without trembles.}

In this paper, we propose a primitive notion of richness in extensive-form games: Extensive-form richness (EF-richness). EF-richness says that for any pure strategy $a$, there is some payoff parameter under which throughout the course of play, following $a$ is always strictly better than switching to anything else. We demonstrate that EF-richness can be satisfied in a class of extensive-form games with perfect recall by suitably choosing payoffs at terminal histories (Lemma 4). We then establish WY’s results in the reduced normal form of these extensive-form games under EF-richness (Theorem 2).

To obtain these results, we first observe that for WY’s results it is both necessary and
sufficient that every rationalizable action is the unique rationalizable action for *some* type (Theorem 1). We call this nonprimitive condition, which is obviously weaker than WY’s richness condition, Richness in Uniquely Rationalizable Actions (RURA). The central step in proving our results is to show that if an extensive-form game satisfies EF-richness, then RURA holds in its reduced normal form (Lemma 6).

To sum up, in this paper we propose the notion of EF-richness which we show enables extending the Weinstein-Yildiz structure theorem to any finite extensive-form game with perfect recall. Specifically we show that for games that satisfy EF-richness we obtain unique rationalizable outcomes for "almost all" types in the universal type space. This implies that such a uniqueness property holds for any refinement. The main implication of our result is that in these extensive-form games, for a prediction to be robust to alternative specifications of beliefs — in the sense proposed by WY and adopted here — it must hold for all rationalizable outcomes.

In the course of this work, we also strengthen one of WY’s result (and hence our extensive-form version). Specifically, WY prove the selection result for finite types: any rationalizable action of any *finite* type \( t \) is the unique rationalizable action of a sequence of finite types converging to \( t \). Ely and Peski (2010) show that finite types are contained in a nongeneric set, so this leaves open the question of whether such a result holds in general. We show that the answer is yes. Namely, the selection result holds no matter \( t \) is finite or infinite.

As mentioned in WY, their approach to robustness of refinements is closely related to that of Fudenberg et al. (1988) (hereafter, FKL) and De kel and Fudenberg (1990) (hereafter, DF), although WY consider simultaneous-move games, while these two studies, like the current paper, prove their main results in the reduced normal form of finite extensive-form games with perfect recall. WY have discussed the differences between their results and those of FKL and DF (see (Weinstein and Yildiz, 2007, pp.390-391)). Here we highlight two critical features of our analysis for dynamic games.

First, instead of pursuing the structure theorem with some version of rationalizability for extensive-form games (see below for more discussion), we prove exactly the same thesis as WY, i.e., a statement about ICR, a normal-form solution concept. Our paper thus belongs

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2We thank Jonathan Weinstein for suggesting this name to us.
to the literature which adopt the normal form to analyze dynamic games. As mentioned above, the normal-form approach is also adopted in FKL and DF’s robustness analysis of dynamic games.

Second, like WY, we assume away any knowledge of payoffs. This enables us to consider a large set of perturbations in which players can entertain any doubt on payoffs. This is analogous to the idea of elaboration with general types in FKL in which players might attach small probability that even their own payoffs are different from the original game. This rich set of perturbations is what drives our conclusion that the tightest robust solution in the normal form is also the tightest that is robust for the extensive form. We therefore formalize the (perhaps by now intuitive) idea that by ruling out any relevant knowledge of the payoff structure, the dynamic structure of extensive-form games offers no force for robust refinements of rationalizability.

An alternative would be to impose "some knowledge of payoffs." That is, one may want to maintain some knowledge assumption even when the beliefs of a given type are perturbed. Intuitively, by allowing a smaller set of perturbations, we may get a tighter solution concept to be robust. An important special case is the elaboration with personal types in FKL and DF in which players know their own payoffs and no play of the opponents can raise doubts in their minds regarding their own payoffs, even though it can raise doubts about anything else. This approach is pursued in an independent work of Penta (2010).

Under EF-richness, Penta (2010) considers multi-stage games with observable actions and obtains generic uniqueness for a refinement of ICR called interim sequential rationalizability (ISR). Unlike ICR, ISR is an extensive-form solution concept based upon initial common belief of sequential rationality. Penta nonetheless shows that ISR, like ICR, is upper hemicontinuous. This may seem to conflict with our claim that no refinement of ICR

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3Many papers in the literature follow Van Neumann and Morganstern claiming that normal form is sufficient to analyze every game. For example, Kohlberg and Mertens (1986) have made forceful arguments that only the reduced normal form of a game should matter or, more precisely, that all extensive-form games with the same reduced normal form should be viewed as strategically equivalent by rational players.

4We note that the distinction of assuming away some versus any knowledge of payoffs plays no role in a static game where no player moves after observing move-related event. One would therefore anticipate that WY’s result for simultaneous-move games holds even when we keep some knowledge of payoffs. Indeed, Penta (2010) proves such a variation. We thank an associate editor for pointing this out to us.

is robust. Indeed together with our results this implies that when all knowledge assumptions are assumed away, ISR coincides with ICR everywhere on the universal type space (Theorem 3).\textsuperscript{6} The intuition behind this equivalence is that without any knowledge, ISR allows players to revise beliefs even about their own payoffs upon being surprised, and under EF-richness this makes every subsequent move "credible" and hence sequential rationality has no bite.\textsuperscript{7}

In contrast, if we maintain some knowledge assumption (e.g., it is common knowledge that players know their own payoffs), ISR is no longer equivalent to ICR. For instance, when we impose the knowledge assumption of own payoffs in the entry deterrence game, the ICR action "fight" is not ISR and can no longer be made uniquely ICR in a perturbed game because the incumbent knows that her own payoffs are those in the original game. This shows that ISR is a solution that depends on whether an aspect of a game is modeled as knowledge (and cannot be unlearned) or belief (and is subject to updating).\textsuperscript{8}

Either approach seems sensible in the abstract, and which one might want to apply would depend on the economic environments in consideration. This is the case even in simultaneous-move games.\textsuperscript{9} For instance, in some applications such as auctions with interdependent values or oligopoly with demand uncertainty, it is conceivable that a player’s opponents have better information regarding the player’s payoffs. For robustness, we may thus consider perturbations in which with small probability players update beliefs about their own payoffs upon seeing an unexpected move of their opponents. However, models with private values are also widespread. Hence, whether or not a type of model is common is not enough to rule out either approach. Here we view the case of assuming away any knowledge of payoffs as a natural benchmark which embodies a stringent robustness check.

\textsuperscript{6}We thank Eddie Dekel for making this observation.

\textsuperscript{7}This intuition is analogous to that in FKL’s results for elaboration with general types.

\textsuperscript{8}To obtain the structure theorem for ISR, Penta also follows FKL and DF to allow for an additional perturbation of payoffs — players know their own payoffs but the payoffs may be slightly different from the original game. To see that this is necessary, consider a complete-information simultaneous-move game in which a player has a weakly dominated rationalizable (and thus ICR and ISR) action. There is no way to perturb the player’s beliefs such that the weakly dominated ISR action becomes the only (sequential) rational action if in the perturbed games the player knows her payoffs are exactly the same as the unperturbed game.

\textsuperscript{9}More generally, some perturbations may seem more natural than others, so in specific context one might want to define robustness differently. For example, the narrower class of perturbations of global games may seem more natural in some environments than the more general class allowed for by WY. See Morris and Shin (2007) for more discussion on this point.
2 The structure theorem under RURA

We first introduce some essential notations and definitions that we adopt from WY. Fix a finite set of players \( N = \{1, 2, ..., n\} \). There is a, possibly unknown, payoff-relevant parameter \( \theta \in \Theta \), where \( \Theta \) is a finite set.\(^{10}\) In this section, we consider only normal-form games. Each player \( i \) has a finite action space \( A_i \) and utility function \( u_i : \Theta \times A \to \mathbb{R} \) (where \( A \equiv \times_{j \in N} A_j \)). Let \( A_{-i} \equiv \times_{j \neq i} A_j \) be the set of all action profiles of player \( i \)'s opponents. By a model, we mean a pair \((\Theta \times T, \kappa)\), where \( T = T_1 \times T_2 \times \cdots \times T_n \) is a compact metric type space associated with beliefs \( \kappa_{t_i} \in \Delta (\Theta \times T_{-i}) \) for each type \( t_i \in T_i \). Assume that \( \Delta (\Theta \times T_{-i}) \) is endowed with the weak* topology and \( t_i \mapsto \kappa_{t_i} \) is a continuous mapping. A finite model is a model such that \(|T| < \infty\).

Given any type \( t_i \) in a model \((\Theta \times T, \kappa)\), we can compute the belief of \( t_i \) on \( \Theta \) by setting \( t_i^1 = \text{ marg}_\Theta \kappa_{t_i} \), which is called the first-order belief of \( t_i \). We can also compute the second-order belief of \( t_i \) (i.e., his belief about \((\theta, t_{-i}^1)\)) by setting

\[
t_i^2 (F) = \kappa_{t_i} \left( \{ (\theta, t_{-i}) : (\theta, t_{-i}^1) \in F \} \right)
\]

for each measurable \( F \subseteq \Theta \times [\Delta (\Theta)]^{n-1} \). We can compute the entire hierarchy of beliefs \((t_i^1, t_i^2, ..., t_i^k, ...)\) by proceeding in this way and write \( h_i (t_i) = (t_i^1, t_i^2, ..., t_i^k, ...) \) for the resulting hierarchy. We denote by \( T^*_i \) the (Mertens-Zamir) universal type space which contains all such belief hierarchies. Let \( T^* = T^*_1 \times \cdots \times T^*_n \) be the set of type profiles and \( T^*_{-i} = \times_{j \neq i} T^*_j \) denote the set of profiles of \( t_{-i} \) for players other than \( i \).

Each \( T^*_i \) is endowed with product topology, so that a sequence of types \((t_i (m))_{m=1}^{\infty}\) converges to a type \( t_i \) (denoted as \( \lim_{m \to \infty} t_i (m) = t_i \)) iff for every \( k \geq 1 \), \( t_i^k (m) \) converges to \( t_i^k \) under weak* topology. Likewise, \( T^* \) is also endowed with product topology, so that a sequence of type profiles \((t (m))_{m=1}^{\infty}\) with \( t (m) = (t_1 (m), ..., t_n (m)) \) converges to a type profile \( t = (t_1, ..., t_n) \) ( \( \lim_{m \to \infty} t (m) = t \)) iff \( t_i (m) \) converges to the type \( t_i \) for each player \( i \).

Mertens and Zamir (1985) show that \( T^*_i \) is a compact metric space, and moreover, there is a homeomorphism \( \kappa^*_i \) between \( T^*_i \) and \( \Delta (\Theta \times T^*_{-i}) \). Hence, \((\Theta \times T^*, \kappa^*)\) is a model where \( \kappa^*_i \equiv \kappa^*_i (t_i) \) for every \( t_i \in T^*_i \). A finite type is some type \( t_i \) in \( T^*_i \) such that \( t_i = h_i (t'_i) \) for some \( t'_i \) in a finite model.

\(^{10}\)WY only assume that \( \Theta \) is a compact metric space. Our structure theorem also holds in this case (see Appendix A.1).
For each \( i \in N \) and for each belief \( \pi \in \Delta (\Theta \times A_{-i}) \), we write \( BR_i(\pi) \) for the set of actions \( a_i \in A_i \) that maximize the expected utility \( u_i(\theta, a_i, a_{-i}) \) under the probability distribution \( \pi \). Following Dekel et al. (2007), we define the solution concept of interim correlated rationalizability (ICR) as follows.

Let \( (\Theta \times T, \kappa) \) be a model. For each player \( i \) and type \( t_i \in T_i \), set \( S_i^0 [t_i] = A_i \), and define sets \( S_i^k [t_i] \) for \( k > 0 \) iteratively by letting \( a_i \in S_i^k [t_i] \) iff \( a_i \in BR_i(\marg_{\Theta \times A_{-i}} \pi) \) for some \( \pi \in \Delta (\Theta \times T_{-i} \times A_{-i}) \) s.t. \( \marg_{\Theta \times T_{-i}} \pi = \kappa_{t_i} \) and \( \pi (\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in S_{-i}^{k-1} [t_{-i}]\}) = 1 \) where \( S_{-i}^{k-1} [t_{-i}] = \Pi_{j \neq i} S_j^{k-1} [t_j] \). Then, define

\[
S_i^\infty [t_i] = \bigcap_{k=0}^{\infty} S_i^k [t_i].
\]

We write \( S^\infty [t] = \prod_{j \in N} S_j^\infty [t_j] \) and \( S_{-i}^\infty [t_{-i}] = \prod_{j \neq i} S_j^\infty [t_j] \). Say that \( \pi \in \Delta (\Theta \times T^*_i \times A_{-i}) \) is valid for a type \( t_i \in T^*_i \) if \( \marg_{\Theta \times T^*_i} \pi = \kappa_{t_i} \) and \( \pi (\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in S_{-i}^\infty [t_{-i}]\}) = 1 \). Dekel et al. (2007) show that \( a_i \in S_i^\infty [t_i] \) iff \( a_i \in BR_i(\marg_{\Theta \times A_{-i}} \pi) \) for some \( \pi \) which is valid for \( t_i \). Moreover, for any \( t_i \) in any model, \( S_i^\infty [t_i] = S_i^\infty [h_i(t_i)] \), i.e., \( S_i^\infty [\cdot] \) only depends on the belief hierarchy of a type.

Following WY, we also define a correspondence \( W^\infty \) which maps each \( t_i \in T_i \) to the set of actions which survive iterated deletion of never-interim-strict best replies for \( t_i \). For each \( i \) and \( t_i \in T_i \), set \( W_i^0 [t_i] = A_i \), and define sets \( W_i^k [t_i] \) for \( k > 0 \) iteratively by letting \( a_i \in W_i^k [t_i] \) iff \( \{a_i\} = BR_i(\marg_{\Theta \times A_{-i}} \pi) \) for some \( \pi \in (\Theta \times T_{-i} \times A_{-i}) \) s.t. \( \marg_{\Theta \times T_{-i}} \pi = \kappa_{t_i} \) and \( \pi (\{(\theta, t_{-i}, a_{-i}) : a_{-i} \in W_{-i}^{k-1} [t_{-i}]\}) = 1 \) where \( W_{-i}^{k-1} [t_{-i}] = \Pi_{j \neq i} W_j^{k-1} [t_j] \). Then, define

\[
W_i^\infty [t_i] = \bigcap_{k=0}^{\infty} W_i^k [t_i].
\]

We now state the richness assumption in WY.

**Definition 1** The game satisfies Richness if for each player \( i \in N \) and each action \( a_i \in A_i \), there exists a payoff parameter \( \theta_{a_i} \in \Theta \) such that

\[
u_i(\theta_{a_i}, a_i, a_{-i}) > \nu_i(\theta_{a_i}, a'_i, a_{-i}), \forall a'_i \neq a_i, \forall a_{-i} \in A_{-i}.
\]

The following notion plays a crucial role in our analysis.
Definition 2  The game satisfies Richness in Uniquely Rationalizable Actions (RURA) if for each player $i$ and each action $a_i$, if $a_i \in S_i^\infty [t_i']$ for some $t_i' \in T_i^*$, then there exists a type $t_i \in T_i^*$ such that $S_i^\infty [t_i] = \{a_i\}$.

Richness implies RURA, as can be seen by choosing any type who assigns probability 1 to $\theta_{a_i}$. The converse is not true. For instance, in the entry deterrence game, payoffs at terminal histories can be assigned so that RURA is satisfied for actions of the incumbent, but there is no way to satisfy Richness. In Section 3, we will generalize this observation by demonstrating that RURA is satisfied in the reduced normal form of any finite extensive-form game with perfect recall and suitably rich payoffs.

A precursor of RURA occurs, in a different and more specialized context, in Frankel et al. (2003). There they note that — in the context of supermodular games — one could replace types that have dominant actions with types that have unique Nash equilibrium actions (which are types with unique rationalizable actions within the class of supermodular games) to start their "global-game infection arguments." Theorem 1 below shows that types with unique rationalizable actions suffice to start infection arguments in the less structured context of WY as they do in the more structured context of global games.\footnote{We thank Stephen Morris for pointing this out to us.} As in WY, Theorem 1 immediately implies the generic uniqueness result stated in Corollary 1.

Theorem 1  RURA holds iff for any type $t_i \in T_i^*$ and action $a_i \in S_i^\infty [t_i]$, there is a sequence of types $(t_i (m))_{m=1}^\infty$ such that $\lim_{m \to \infty} t_i (m) = t_i$ and $S_i^\infty [t_i (m)] = \{a_i\}$, $\forall m$.

Corollary 1  Under RURA, $\{t \in T^* : |S^\infty [t]| = 1\}$ is open and dense in product topology.

Observe that the "if" part of Theorem 1 is immediate: if an action is never the unique rationalizable action of any type, the sequence in Theorem 1 cannot exist. The "only if" part of Theorem 1 strengthens WY’s Proposition 1 in two ways. First, it replaces Richness with RURA which is weaker and in fact equivalent to the structure theorem. This will be useful for our analysis of dynamic games in Section 3. Second, in WY’s Proposition 1, $t_i \in T_i^*$ is a finite type, whereas in Theorem 1 $t_i$ can be finite or infinite. Infinite types are necessary to model the information structure in global games as well as other applications and recall that
WY use the "only if" part of Theorem 1 to conclude that any refinement of rationalizability is not robust. In Section 4, we will use the "only if" part (and in particular that it works for any type) to prove that ICR is equal to ISR defined in Penta (2010) everywhere on the universal type space.

The proof of the "only if" part requires three lemmas. First, Lemmas 1 and 2 are the counterparts of Lemmas 6 and 7 of WY under RURA. Theirs proofs follow closely the ideas of Lemmas 6 and 7 of WY and are presented in an Online Appendix linked to the author’s website.

**Lemma 1** Under RURA, for any finite type \( t_i \in T_i^* \) and action \( a_i \in S_i^\infty [t_i] \), there exists a sequence of finite models \( ((\Theta \times T^m, \kappa^m))_{m=1}^\infty \) and a sequence of finite types \( (t_i (m))_{m=1}^\infty \) such that \( t_i (m) \in T_i^m \) and \( a_i \in W_i^\infty [t_i (m)] \) for all \( m \) and \( \lim_{m \to \infty} t_i (m) = t_i \).

**Lemma 2** Under RURA, for any finite type \( t_i \in T_i^* \), action \( a_i \in W_i^\infty [t_i] \), and integer \( k \geq 0 \), there exists a finite type \( \bar{t}_i \) such that \( \bar{t}_i^k = t_i^k \) for all \( k' \leq k \) and \( S_i^\infty [\bar{t}_i] = \{ a_i \} \).

**Lemma 3** For any type \( t_i \in T_i^* \), there is a sequence of finite type \( (t_i (m))_{m=1}^\infty \) such that \( S_i^\infty [t_i (m)] = S_i^\infty [t_i] \) and \( \lim_{m \to \infty} t_i (m) = t_i \).

Lemma 3 is trivial when \( t_i \) is finite. When \( t_i \) is infinite, it shows that there is a sequence of finite types converging to \( t_i \) in product topology, and moreover, none of rationalizable action(s) of \( t_i \) is lost along the sequence. With Lemma 3, the result with an infinite \( t_i \) in Theorem 1 is implied by the result with finite types (by applying Theorem 1 to every finite type \( t_i (m) \) in Lemma 3 and taking an appropriate diagonal sequence). The intuition of Lemma 3 follows from two existing results: finite types are dense in product topology (Mertens and Zamir, 1985, Theorem 3.1), and finite types are dense in strategic topology (Dekel et al., 2006, Theorem 3).12 Indeed, the proof of Lemma 3 which can be found in

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12Strategic topology on types is the coarsest topology in which rationalizable behaviors are continuous. More precisely, (Dekel et al., 2006, Theorem 3) shows that for any type \( t_i \), any finite game \( G \), any \( \varepsilon > 0 \), there is a finite type \( t'_i \) such that every rationalizable action of \( t_i \) in \( G \) is \( \varepsilon \)-rationalizable for \( t'_i \). Here we can replace \( \varepsilon \)-rationalizable with \((0-)\)-rationalizable because we consider a fixed finite game (see (Dekel et al., 2006, Lemma 12)).
Appendix A.2 combines the ideas in the proofs of these two results.

3 The structure theorem under EF-richness

In this section, we demonstrate that RURA can be satisfied in the reduced normal form of any finite extensive-form game with perfect recall and sufficiently rich payoffs. To motivate our setup and results, consider Example 1 in Figure 1. In this game, the two pure strategies $D_1C_3$ and $D_1D_3$ of player 1 are obviously equivalent: the history $C_1C_2$ has been precluded by either $D_1C_3$ or $D_1D_3$ and hence $D_1C_3$ and $D_1D_3$ induce the same outcome no matter what player 2 chooses. If we treat $D_1C_3$ and $D_1D_3$ as different strategies, for any possible payoffs and belief, player 1 cannot have either $D_1C_3$ or $D_1D_3$ as the unique best reply. Thus, RURA is not applicable.

One solution to this problem is to define the reduced form by adopting a singleton selection from such equivalent extensive-form pure strategies. This idea is suggested by WY and dates back to the early works by Kuhn (1950, 1953). In the example it means that we should select either $D_1C_3$ or $D_1D_3$ in defining the reduced form. Similar to this we will just "ignore" the part of a strategy that a player’s own strategy precludes. Specifically, our notion of Extensive-form richness says that for any pure strategy $a_i$, there is some payoff parameter $\theta_{ai}$ such that throughout the course of play, following $a$ is always strictly better than switching to anything else.

To formalize the idea, we introduce some notations and definitions from (Osborne and Rubinstein, 1994, Chapters 6 and 11). Let $\Gamma = (N, H, (I_i)_{i \in N})$ be a finite extensive game form with perfect recall: $H$ is a finite set of histories, and $I_i$ is player $i$’s information partition on $H_i$ where $H_i$ is the set of histories after which player $i$ moves. For any history $h \in H_i$, let $I_i(h) \in I_i$ be the information set which contains $h$ and $B_i(h) = \{b_i : (h, b_i) \in H\}$ be the set of actions available to player $i$ at $h$. It is required that $B_i(\overline{h}) = B_i(h)$ for any $\overline{h} \in I_i(h)$. Let $Z$ be the set of terminal histories and $U_i : \Theta \times Z \to \mathbb{R}$ be the payoff function of player $i$. Then, $(\Gamma, (U_i)_{i \in N})$ is a finite extensive-form game.

A pure strategy is a mapping $a_i : H_i \rightarrow \bigcup_{h \in H_i} B_i(h)$ such that for any $h \in H_i$, $a_i(h) \in B_i(h)$ and $a_i(\overline{h}) = a_i(h)$ for all $\overline{h} \in I_i(h)$. For any pure strategy $a_i$, let $H(a_i)$ be the
set of all the histories in which all the actions of player \( i \) are those dictated by \( a_i \), i.e., \( h \in H (a_i) \) iff for any proper subhistory \( h' \in H_i \) of \( h \), \( (h', a_i(h')) \) is also a subhistory of \( h \). Let \( A_i(h) \) be the set of player \( i \)'s pure strategies consistent with \( h \), i.e., \( a_i \in A_i(h) \) iff \( h \in H (a_i) \). Let \( A_{-i}(h) = \times_{j \neq i} A_j(h) \). For each pure strategy profile \( a \), denote by \( z(a) \) the unique terminal history induced by \( a \). Say a pure strategy profile \( a \) reaches an information set \( I_i(h) \) if \( I_i(h) \) contains a subhistory of \( z(a) \). Finally, two pure strategies \( a_i \) and \( a'_i \) are said to be equivalent if \( z(a_i, a_{-i}) = z(a'_i, a_{-i}) \) for any player \( i \)'s opponents' pure strategy profile \( a_{-i} \). Let \( G^{U} \equiv (N, (A_i)_{i \in N}, (u_i)_{i \in N}) \) be the reduced normal-form representation of \( (\Gamma, (U_i)_{i \in N}) \). We will apply the notions and results in the previous section to study \( G^{U} \).

**Definition 3** Say \( (\Gamma, (U_i)_{i \in N}) \) satisfies Extensive-form Richness (EF-richness) if for any \( i \in N \) and any \( a_i \in A_i \), there is some \( \theta_{a_i} \in \Theta \) such that for any \( a'_i \in A_i \) and any \( a_{-i} \in A_{-i} \), if \( a'_i(h') \neq a_i(h') \) for some subhistory \( h' \in H_i \) of \( z(a_i, a_{-i}) \), then \( U_i(\theta_{a_i}, z(a_i, a_{-i})) > U_i(\theta_{a_i}, z(a'_i, a_{-i})) \).

For instance, to satisfy EF-richness for the pure strategies C1C3 of player 1 and C2 of player 2, the payoffs under \( \theta_{C1C3} \) and \( \theta_{C2} \) can be assigned as in Figure 1. Lemma 4 below states that EF-richness can be satisfied in a class of extensive-form games with perfect recall by suitably choosing payoffs at terminal histories. The proof of Lemma 4 is similar to specifying payoffs of "crazy types" in (Fudenberg et al., 1988, p.366) and can be found in Appendix A.4. Lemma 5 is Theorem 1 of Kuhn (1953) and will be used later.

**Lemma 4** There are some set of payoff parameters \( \Theta \) and utility functions \( (U_i)_{i \in N} \) with \( U_i : \Theta \times Z \rightarrow \mathbb{R} \) such that \( (\Gamma, (U_i)_{i \in N}) \) satisfies EF-richness.

**Lemma 5** Two pure strategies \( a_i \) and \( a'_i \) in \( \Gamma \) are equivalent if \( a'_i(h) = a_i(h), \forall h \in H(a_i) \).

We now present our main result.

**Theorem 2** Suppose that the extensive-form game \( (\Gamma, (U_i)_{i \in N}) \) satisfies EF-richness. Then, for any type \( t_i \) and any pure strategy \( a_i \in S_i^{\infty}[t_i] \) in \( G^{U} \), there is a sequence of types \( (t_i(m))_{m=1}^{\infty} \) such that \( \lim_{m \rightarrow \infty} t_i(m) = t_i \) and \( S_i^{\infty}[t_i(m)] = \{a_i\} \) for any \( m \). Moreover, \( \{t \in T^* : |S^{\infty}[t]| = 1\} \) is open and dense in product topology.
Theorem 2 immediately follows from Theorem 1 and the Lemma 6 below. Lemma 6 shows that RURA holds in the reduced normal-form representation $G^\Gamma U$ whenever the extensive-form game $(\Gamma, (U_i)_{i \in \mathcal{N}})$ satisfies EF-richness.

**Lemma 6** Suppose that the extensive-form game $(\Gamma, (U_i)_{i \in \mathcal{N}})$ satisfies EF-richness. Then, for any pure strategy $a_i$ in $G^\Gamma U$, there is a finite type $t_i \in T_i^*$ such that $S_i^\infty [t_i] = \{a_i\}$. Hence, RURA holds in $G^\Gamma U$.

To prove this lemma, we define a finite model $(\Theta \times T, \kappa)$ as follows.

\[
T_i = \{ t_i.a_i : a_i \in A_i \}, \forall i \in \mathcal{N} \tag{\ast}
\]

\[
\kappa_{t_i.a_i} (\{(\theta, t_{-i})\}) = \begin{cases} \frac{1}{|T_{-i}|}, & \text{if } \theta = \theta_{a_i}; \\ 0, & \text{otherwise}, \end{cases} \quad \forall (\theta, t_{-i}) \in \Theta \times T_{-i}, \forall i \in \mathcal{N}.
\]

Note that in this model, for every $a_i \in A_i$, player $i$ has a type $t_i.a_i$ which assigns probability one to $\theta_{a_i}$ and positive probability to every $t_{-i,a_{-i}}$ (where $t_{-i,a_{-i}} \equiv (t_{j,a_j})_{j \neq i}$ for any $a_{-i} = (a_j)_{j \neq i}$), and moreover, these are common belief among all players. The claim below together with Lemma 5 then implies that $S_i^\infty [t_{i,a_i}] = \{a_i\}$. Since $S_i^\infty [\cdot]$ only depends on the belief hierarchy of a type, there is a finite type $t_i \in T_i^*$ such that $S_i^\infty [t_i] = \{a_i\}$.

The key idea to prove the claim is to make all information sets reached with positive probability. The proof proceeds by induction on the lengths of histories: the initial step follows because the initial history is trivially reached and $t_i.a_i$ assigns probability one to $\theta_{a_i}$. For the induction step, let $a'_i$ be a pure strategy such that $a'_i(h) \neq a_i(h)$ for some history $h$ in $H(a_i) \cap H_i$ with length $m + 1$. Since $t_i.a_i$ assigns positive probability to every $t_{-i,a_{-i}}$, perfect recall and the induction hypothesis ensure that $I_i(h)$ is reached with positive probability under any valid belief $\pi$ of $t_{i,a_i}$. Moreover, since $t_i.a_i$ assigns probability one to $\theta_{a_i}$, $a_i$ generates a strictly higher expected utility than $a'_i$ under $\pi$ and thus $a'_i \notin S_i^\infty [t_{i,a_i}]$.

**Claim 1** For any $a_i, a'_i \in A_i$, $a'_i \in S_i^\infty [t_{i,a_i}] \Rightarrow a'_i(h) = a_i(h), \forall h \in H(a_i) \cap H_i$.

**Proof.** We prove this claim by induction. First, consider player $i$ who moves at the initial history $h$. Since the initial history is a subhistory of any history and $t_{i,a_i}$ assigns probability one
to \(\theta_{a_i}\), for any pure strategy \(a'_i\) with \(a'_i(\varepsilon) \neq a_i(\varepsilon)\), \(U_i(\theta_{a_i}, z(a_i, a_{-i})) > U_i(\theta_{a_i}, z(a'_i, a_{-i}))\) for any \(a_{-i} \in A_{-i}\). Hence, \(a'_i \in S^\infty_{t,a_i}\) only if \(a'_i(\varepsilon) = a_i(\varepsilon)\).

Now suppose that our claim holds for every player \(j\) and every history with length less than or equal to \(m\) where \(m \geq 0\), i.e.,

\[
a'_j \in S^\infty_{j} \left[ t_{j,a_j} \right] \Rightarrow a'_j(h') = a_j(h') \quad \forall h' \in H_j(a_j) \cap H_j \text{ with length } \leq m. \tag{IH}
\]

Fix a history \(h \in H(a_i) \cap H_i\) with length \(m + 1\). We now show that under (IH) any pure strategy \(a'_i\) with \(a'_i(h) \neq a_i(h)\) cannot be rationalizable for \(t_{i,a_i}\) because for any \(\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})\) which is valid for \(t_{i,a_i}\),

\[
\sum_{(\theta,t_{-i},a_{-i}) \in \Theta \times T_{-i} \times A_{-i}} [u_i(\theta, a_i, a_{-i}) - u_i(\theta, a'_i, a_{-i})] \pi(\{\theta, t_{-i}, a_{-i}\}) > 0. \tag{\bigstar}
\]

We prove (\bigstar) in two steps:

**Step 1** Under (IH), for any \(\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})\) which is valid for \(t_{i,a_i}\), \(\pi(\{a_{-i}\}) > 0\) for some action \(a_{-i} \in A_{-i}\) such that \(h\) is a subhistory of \(z(a_i, a_{-i})\).

Recall that for any history \(h'\), \(a_{-i} = (a_j)_{j \neq i} \in A_{-i}(h')\) iff \(h' \in H(a_j)\) for all \(j \neq i\). Then, \(A_{-i}(h) \neq \emptyset\) because perfect recall ensures that no subhistories of \(h\) are contained in the same information set of a player. Let \(a''_{-i} = (a_j)_{j \neq i} \in A_{-i}(h)\). By (IH), \(a''_j \in S^\infty_{j} \left[ t_{j,a'_j} \right]\) implies that \(a''_j(h') = a'_j(h')\), for any \(h' \in H_j(a_j) \cap H_j\) which is a proper subhistory of \(h\). Then, since \(\kappa_{t_{i,a_i}}(\{t_{-i},a''_{-i}\}) > 0\) and \(\pi\) is valid for \(t_{i,a_i}\), \(\pi(\{a_{-i}\}) > 0\) for some \(a_{-i} \in A_{-i}(h)\). Moreover, since \(a_{-i} \in A_{-i}(h)\) and \(h \in H(a_i), h \in H(a_j)\) for all \(j \in N\). Therefore, \(h\) is a subhistory of \(z(a_i, a_{-i})\).

**Step 2** Under (IH), any pure strategy \(a'_i\) with \(a'_i(h) \neq a_i(h)\) is not rationalizable for \(t_{i,a_i}\).

Suppose that \(\pi \in \Delta(\Theta \times T_{-i} \times A_{-i})\) is valid for \(t_{i,a_i}\). Since \(t_{i,a_i}\) assigns probability one to \(\theta_{a_i}\), we can replace \(\theta\) by \(\theta_{a_i}\) in the summand of (\bigstar). Let \(a_{-i} \in A_{-i}\) and there are two cases: (i) if \(a'_i(h') \neq a_i(h')\) for some subhistory \(h' \in H_i\) of \(z(a_i, a_{-i})\), then \(u_i(\theta_{a_i}, a_i, a_{-i}) > u_i(\theta_{a_i}, a'_i, a_{-i})\) by the definition of \(\theta_{a_i}\); (ii) if \(a'_i(h') = a_i(h')\) for any subhistory \(h' \in H_i\) of \(z(a_i, a_{-i})\), then \(z(a_i, a_{-i}) = z(a'_i, a_{-i})\) and hence \(u_i(\theta_{a_i}, a_i, a_{-i}) = u_i(\theta_{a_i}, a'_i, a_{-i})\). By step
1, case (i) happens with positive probability under \( \pi \). Therefore, \( (\dagger) \) holds and \( a'_i \) is not rationalizable for \( t_{i,a_i} \).

We conclude the section by elaborating the differences between the arguments in FKL and DF (see FKL’s Theorem 3 and DF’s Proposition 4.2) and the proof of Lemma 6. Both FKL and DF start with a complete information game in which payoffs are common knowledge. A player with the information and payoffs in this complete information game is called a "sane type." A perturbed game is an incomplete information game where with small probability a player becomes a "crazy type" whose payoffs and information drastically differ from her sane type. FKL show for instance that every Nash equilibrium profile in the complete information game can be approximated by some strict Nash equilibrium profile in the perturbed game.

To facilitate comparison, here types with multiple rationalizable actions can be viewed as sane types, whereas types with unique rationalizable actions can be viewed as crazy types. Our proof of Lemma 6 shares the feature of FKL’s and DF’s proofs that an incomplete information game with crazy types is constructed such that every information set is reached with positive probability. However, there is no sane type in the model constructed in \( (\star) \). This is because Lemma 6 is only one step toward the structure theorem. Sane types appear when we combine Lemma 6 and Theorem 1 to prove that crazy types are dense.

For instance, consider a rationalizable action \( a_i \) of a finite type \( t_i \) for simplicity. In proving Theorem 1, we first use Lemma 1 to make \( a_i \) survive the iterated deletion of never-strict best replies for some type \( t_{i,0}^{a_i} \) which is close to \( t_i \). The crazy types constructed in Lemma 6 are then used to start the following infection. First, the belief \( \pi \) under which \( a_i \) is a strict best reply for \( t_{i,0}^{a_i} \) together with the mapping \((\theta, t_{-i}, a_{-i}) \mapsto (\theta, t_{-i,a_{-i}})\) identifies a crazy type \( t_{i,1}^{a_i} \) who has the same first-order belief as \( t_{i,0}^{a_i} \) and plays \( a_i \). Second, \( \pi \) and the mapping \((\theta, t_{-i}, a_{-i}) \mapsto (\theta, t_{-i,1}^{a_i})\) identifies another crazy type \( t_{i,2}^{a_i} \) who has the same the second-order belief as \( t_{i,0}^{a_i} \) and plays \( a_i \), and so on. In this way we produce more and more crazy types eventually approaching every sane type. The perturbed information structure is defined recursively. Thus, this construction is close to WY but differs from FKL and DF.

Another difference is that in proving their results, FKL (and similarly DF) aim to show that some profile constitutes a strict Nash equilibrium, whereas in Lemma 6 we show that the strategy \( a_i \) is uniquely rationalizable for the crazy type \( t_{i,a_i} \). In particular, the profile
(s_i)_{i \in N} with s_i [t_{i,a_i}] = a_i clearly constitutes a strict Nash equilibrium in the model defined in (★): given s_{-i}, every h in H (a_i) occurs with positive probability and hence a_i does strictly better than any other a'_i for t_{i,a_i}. Here without fixing the strategy of t_{i,a_j} being a_j, we can only know that every h in H (a_i) occurs with positive probability by proving inductively that a rationalizable strategy for t_{i,a_i} must coincide with a_i along histories in H (a_i) \cap H_i.

4 Equivalence between ICR and ISR

Penta (2010) considers a finite multi-stage game with observable actions. In our notations, let (\Gamma, (U_i)_{i \in N}) = (N, H, (I_i)_{i \in N}, (U_i)_{i \in N}) be a multistage game with observable action and G^{\Gamma,U} be the reduced normal-form representation of (\Gamma, (U_i)_{i \in N}).

Let (\Theta \times T, \kappa) be a model. A conditional probability system \pi on \Theta \times T_{-i} \times A_{-i} specifies for every h \in H \setminus Z some \pi (\cdot | h) \in \Delta (\Theta \times T_{-i} \times A_{-i} (h)) and \pi is consistent with Bayes’ rule whenever possible. Say a_i \in A_i is a sequential best reply to a conditional probability system \pi on \Theta \times T_{-i} \times A_{-i} iff

\[ a_i \in \arg \max_{a'_i \in A_i(h)} \int_{\Theta \times T_{-i} \times A_{-i}(h)} u_i (a'_i, a_{-i}, \theta) d\pi (\theta, t_{-i}, a_{-i} | h), \forall h \in H (a_i). \]

Penta (2010) defines the solution concept of Interim Sequential Rationalizability (ISR) as follows. For each player i and type t_i \in T_i, set ISR^{0}_i [t_i] = A_i, and define sets ISR^{k}_i [t_i] for k > 0 iteratively by letting action a_i \in ISR^{k}_i [t_i] iff a_i is a sequential best reply to a conditional probability system \pi on \Theta \times T_{-i} \times A_{-i} s.t. \marg_{\Theta \times T_i} \pi (\cdot | \phi) = \kappa_{t_i} and \pi \left( \{(\theta, t_{-i}, a_{-i}) : a_{-i} \in ISR^{k-1}_{-i} [t_{-i}] \} | \phi \right) = 1 where \phi is the initial history and ISR^{k-1}_{-i} [t_{-i}] = \Pi_{j \neq i} ISR^{k-1}_j [t_j]. Then, define ISR^{\infty}_i [t_i] = \bigcap_{k=0}^{\infty} ISR^k_i [t_i].

As we mentioned in the introduction, in the models which assume away any knowledge of payoffs (that we study here), ISR allows players to revise beliefs about their own payoffs upon being surprised. More precisely, the only restriction that ISR puts on the beliefs held at zero-probability histories comes from the definition of a model. This means that once surprised, type t_i may assign positive probability to pairs (\theta, t_{-i}) that were initially given zero probability by the belief \kappa_{t_i}. Under EF-richness, this makes every subsequent
move "credible" and hence sequential rationality has no bite. The idea is formalized in the following result.

**Theorem 3**  *If the extensive-form game* $(\Gamma, (U_i)_{i \in N})$ *satisfies EF-richness, then* $\text{ISR}^\infty_i [t_i] = S^\infty_i [t_i]$ *for all* $t_i \in T_i^*$.

Theorem 3 is a direct consequence of Theorem 1 and the following proposition due to Penta (2010). See Appendix A.5 for a proof.

**Proposition 1 (Penta (2010))** $\text{ISR}^\infty_i [\cdot]$ *is nonempty and upper hemicontinuous.*

## 5 Concluding remarks

We conclude with some remarks on other related works. Firstly, Weinstein and Yildiz (2010b) have further extended our result to prove a structure theorem for ICR in the normal form of infinite-horizon games. Weinstein and Yildiz apply this structure theorem for infinite-horizon games to study important applications such as the Rubinstein bargaining model and repeated games. Secondly, Oury and Tercieux (2009) assume a weaker richness condition and use a version of WY’s argument in their study of robust implementation.

Two other recent papers are closely related to Weinstein and Yildiz (2007). Neither of them assumes any richness condition and neither of them deals with dynamic games. First, Weinstein and Yildiz (2010a) characterize, for any fixed finite order, the set of actions that can be played in a Bayesian Nash equilibrium by some type whose lower-order beliefs are all as in the original type in nice games (where action spaces are compact intervals, utilities continuous and strictly concave in own action). Second, in a fixed game, the rationalizable correspondence is continuous in product topology around types with unique rationalizable actions. Ely and Peski (2010) generalize this observation to define a type to be regular if it exhibits a similar notion of continuity in *every* finite game. They show that regular types are generic in the universal type space.
A Appendix

A.1 Compact metric Θ

Here we sketch how to extend our result when Θ is a compact metric space as in WY. This includes the case that Θ = Θ₁ × ⋯ × Θₙ and Θᵢ = [0, 1]^[Z]. Our arguments for the "only if" part of Theorem 1 (which is the direction we use in proving Theorem 2) still go through if we strengthen RURA as follows (see the Online Appendix for details).

**RURA'** There is a finite set of payoff parameters \(\overline{\Theta} \subset \Theta\) such that for each player \(i\) and each action \(a_i\), if \(a_i \in \mathcal{S}_i^\infty [t_i']\) for some type \(t_i' \in T_i^*\), then there exists a finite type \(t_i \in T_i^*\) such that \(\mathcal{S}_i^\infty [t_i] = \{a_i\}\) and \(\kappa_t [\overline{\Theta}] = 1\).

We then follow WY to assume that every utility function \(u_i\) is continuous and note that a finite model is now a model \((\Theta' \times T, \kappa)\) such that \(\Theta' \subset \Theta\) and \(|\Theta' \times T| < \infty\). To see that EF-richness implies RURA', let \(\overline{\Theta} = \{\theta_a : a_i \in A_i\text{ and } i \in N\}\) and apply the proof of Lemma 6.

A.2 Proof of Lemma 3

In this proof, we only require that Θ is a compact metric space equipped with metric \(d^0\). Let \(j \in \{1, 2, ..., n\}\) denote a generic player. Recall that the universal type space \(T_j^*\) endowed with product topology is a compact metric space. In the proof, let \(d_j\) be the metric on \(T_j^*\) induced from the Prohorov metric.\(^{13}\) Specifically, for any \(t_j, t'_j \in T_j^*\), let \(d_j^1 (t_j^1, t'_j^1)\) be the Prohorov distance between \(t_j^1\) and \(t'_j^1\) (recall \(t_j^1, t'_j^1 \in \Delta (\Theta)\)). Recursively, for any integer \(k \geq 2\), and \(t_j, t'_j \in T_j^*\), let \(d_j^k (t_j^k, t'_j^k)\) be the Prohorov distance between \(t_j^k\) and \(t'_j^k\) where \(t_j^k, t'_j^k \in \Delta (\Theta)\).

\(^{13}\)Let \(Y\) be an arbitrary compact metric space endowed with metric \(\rho\) and the Borel σ-algebra. For any two \(\mu, \mu' \in \Delta (Y)\), the Prohorov distance between \(\mu\) and \(\mu'\) is defined as

\[
d (\mu, \mu') = \inf \{\varepsilon > 0 : \mu (E) \leq \mu' (E^\varepsilon) + \varepsilon \text{ for all Borel set } E \subseteq Y\}
\]

where \(A^\varepsilon \equiv \{y \in Y : \inf_{y' \in E} \rho (y, y') < \varepsilon\}\). It is known that the Prohorov metric metrizes the weak*-topology on \(\Delta (Y)\) (see (Dudley, 2002, 11.3.3. Theorem)).
$\Delta (\Theta \times T^{k-1}_{j})$ in which $T^{k-1}_{j}$ is the space of all $(k - 1)^{th}$-order beliefs of player $j$’s opponents and $\Theta \times T^{k-1}_{j}$ is equipped with the metric $\rho^{k-1}_{j}$ defined as $\rho^{k-1}_{j}((\theta, t^{k-1}_{j}), (\theta', t'^{k-1}_{j})) \equiv \max (d^{0}(\theta, \theta'), \max_{j' \neq j} d^{k-1}_{j'}(t'_{j'}, t'_{j}))$. Let $d_{j}(t_{j}, t'_{j}) \equiv \sum_{k=1}^{\infty} 2^{-k}d^{k}_{j}(t^{k}_{j}, t'^{k}_{j})$, i.e., $d_{j}$ is the product metric which metrizes the product topology on $T^{*}_{j}$.

**Lemma 3** For any type $\bar{t}_{i} \in T^{*}_{i}$, there is a sequence of finite types $(t_{i}(m))_{m=1}^{\infty}$ such that $S_{i}^{\infty}[t_{i}(m)] = S_{i}^{\infty}[\bar{t}_{i}]$ for all $m$ and $\lim_{m \to \infty} t_{i}(m) = \bar{t}_{i}$.

**Proof.** We divide the proof into three steps.

**Step 1. Construct the sequence of finite types.**

Since $T^{*}_{j}$ is a compact metric space, for each natural number $m$, $T^{*}_{j}$ can be covered by finitely many open balls with radius $1/2m$. Let $T^{*}_{j}$ be the finite measurable partition $T_{j,m}$ induced from these open balls and thus for any $T_{j} \in T_{j,m}$, and $t_{j}$ and $t'_{j}$ in $T_{j}$, $d_{j}(t_{j}, t'_{j}) < 1/m$. Second, let $T_{j,0}$ be the finite measurable partition induced by rationalizable sets, i.e., for any $T_{j} \in T_{j,0}$, $t_{j}, t'_{j} \in T_{j}$ iff $S_{j}^{\infty}[t_{j}] = S_{j}^{\infty}[t'_{j}]$. Let $\bar{T}_{j,m}$ be the join (coarsest common refinement) of $T_{j,0}$ and $T_{j,m}$. Let $f_{j,m} : T^{*}_{j} \to \bar{T}_{j,m}$ be a mapping such that $f_{j,m}(t_{j}) = \bar{t}_{j,m}$ iff $t_{j} \in \bar{T}_{j,m}$. Moreover, for each $\bar{t}_{j,m} \subseteq \bar{T}_{j,m}$, select arbitrarily a type $t_{j,m} \in \bar{t}_{j,m}$. It follows that

$$d_{j}(t_{j}, t_{j,m}) < 1/m, \forall t_{j} \in \bar{t}_{j,m}. \tag{1}$$

Define a sequence of finite models $\left(\left(\Theta \times \bar{T}^{m}_{j}, \bar{\kappa}^{m}_{j}\right)\right)_{m=1}^{\infty}$ by letting $\bar{T}_{j}^{m} \equiv \bar{T}_{j,m}$, and for each $\bar{t}_{j,m} \in \bar{T}_{j}^{m}$,

$$\bar{\kappa}_{j}^{m}_{t_{j,m}} \equiv \kappa_{j}^{m}_{t_{j,m}} \left[\{ (\theta, t_{j}) : f_{j,m}(t_{j}) = \bar{t}_{j,m} \}\right], \forall (\theta, t_{j}) \in \Theta \times \bar{T}_{j}^{m}. \tag{2}$$

Note $\bar{t}_{j,m}$ denotes both a subset of $T^{*}_{j}$ and a type in the model $\left(\Theta \times \bar{T}^{m}_{j}, \bar{\kappa}^{m}_{j}\right)$. We will write $\bar{t}_{j,m} \in \bar{T}_{j,m}$ for the former and $\bar{t}_{j,m} \in \bar{T}_{j}^{m}$ for the latter when necessary. Let $\bar{t}_{i}(m) \equiv f_{i,m}(\bar{t}_{i})$ for every $m$. Step 2 and Step 3 below show that $\lim_{m \to \infty} \bar{t}_{i}(m) = \bar{t}_{i}$ and $S_{i}^{\infty}[\bar{t}_{i}(m)] \supseteq S_{i}^{\infty}[\bar{t}_{i}]$ for all $m$. Since $S_{i}^{\infty}[\cdot]$ is upper hemicontinuous and $\lim_{m \to \infty} \bar{t}_{i}(m) = \bar{t}_{i}$, it follows

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<sup>14</sup>Measurability follows from upper hemicontinuity (u.h.c.) of $S_{i}^{\infty}[\cdot]$: If $A'_{j} \subseteq A_{j}$ is 1−minimal in the sense that there is no type $t_{j}$ with $S_{j}^{\infty}[t_{j}] \subseteq A'_{j}$, then u.h.c. implies $\{ t_{j} : S_{j}^{\infty}[t_{j}] = A'_{j} \} = \{ t_{j} : S_{j}^{\infty}[t_{j}] \subseteq A'_{j} \}$ is open and hence measurable; if $A'_{j} \subseteq A_{j}$ is 2−minimal in the sense that $S_{j}^{\infty}[t_{j}] \subseteq A'_{j}$ iff $S_{j}^{\infty}[t_{j}]$ is 1−minimal then $\{ t_{j} : S_{j}^{\infty}[t_{j}] = A'_{j} \} = \{ t_{j} : S_{j}^{\infty}[t_{j}] \subseteq A'_{j} \} \setminus \{ t_{j} : S_{j}^{\infty}[t_{j}]$ is 1−minimal $\}$ is measurable, and so on. Since $A_{j}$ is a finite set, every $S_{j}^{\infty}[t_{j}]$ is $k$−minimal for some $k$ and thus $\{ t_{j} : S_{j}^{\infty}[t_{j}] = A'_{j} \}$ is measurable, $\forall A'_{j} \subseteq A_{j}$.
that \( S_i^\infty \mathbf{[} t_i (m) \mathbf{]} = S_i^\infty \mathbf{[} t_i \mathbf{]} \) for sufficiently large \( m \), say \( m \geq \overline{m} \). We then define \( \bar{t}_i (m) = t_i (m + \overline{m}) \forall m \) and \( (\bar{t}_i (m))_{m=1}^\infty \) is the desired sequence.

**Step 2.** For each \( m \) and each \( t_j \in T_j^* \), \( S_j^\infty \mathbf{[} f_{j,m} (t_j) \mathbf{]} \supseteq S_j^\infty \mathbf{[} t_j \mathbf{]} \).

First, for each \( \bar{t}_{j,m} \in \tilde{T}_m^j \), we define \( \overline{S}_j [\bar{t}_{j,m}] = S_j^\infty [t_{j,m}] \). We show that \( \overline{S}_j [\cdot] \) satisfies the best-reply property on the model \( (\Theta \times \tilde{T}_m^j, \overline{\kappa}^m) \) (see (Dekel et al., 2007, Definition 1)). To see this, suppose that \( a_j \in \overline{S}_j [\bar{t}_{j,m}] \). Since \( \overline{S}_j [\bar{t}_{j,m}] = S_j^\infty [t_{j,m}] \), \( a_j \in S_j^\infty [t_{j,m}] \). Thus, \( a_j \in BR_j \left( \text{marg}_{\Theta \times A_{-j}} \pi \right) \) for some \( \pi \in \Delta (\Theta \times T_{-j}^* \times A_{-j}) \) which is valid for \( t_{j,m} \).

Define \( \bar{\pi} \in \Delta \left( \Theta \times \tilde{T}_m^j \times A_{-j} \right) \) such that

\[
\bar{\pi} \left( (\theta, \bar{t}_{-j,m}, a_{-j}) \right) \equiv \pi \left( \{ (\theta, t_{-j}, a_{-j}) : f_{-j,m} (t_{-j}) = \bar{t}_{-j,m} \} \right), \forall (\theta, \bar{t}_{-j,m}, a_{-j}) \tag{3}
\]

Since \( \pi \) is valid for \( t_{j,m} \), \( \text{marg}_{\Theta \times T_{-j}^*} \pi = \kappa_{t_{j,m}}^\infty \). Hence, by (2), \( \text{marg}_{\Theta \times \tilde{T}_m^j} \bar{\pi} = \kappa_{\bar{t}_{j,m}}^\infty \). Moreover,

\[
\bar{\pi} \left( \{ (\theta, \bar{t}_{-j,m}, a_{-j}) : a_{-j} \in \overline{S}_j [\bar{t}_{-j,m}] \} \right) \\
= \pi \left( \{ (\theta, t_{-j}, a_{-j}) : f_{-j,m} (t_{-j}) = \bar{t}_{-j,m} \text{ and } a_{-j} \in \overline{S}_j [\bar{t}_{-j,m}] \} \right) \\
= \pi \left( \{ (\theta, t_{-j}, a_{-j}) : f_{-j,m} (t_{-j}) = \bar{t}_{-j,m} \text{ and } a_{-j} \in S_j^\infty [t_{j,m}] \} \right) \\
= \pi \left( \{ (\theta, t_{-j}, a_{-j}) : a_{-j} \in S_j^\infty [t_{-j}] \} \right) \\
= 1
\]

where the first equality follows from (3), the second follows because \( \overline{S}_j [\bar{t}_{-j,m}] = S_j^\infty [t_{-j,m}] \), the third follows because every \( t_{-j} \in \bar{t}_{-j,m} \) has the same rationalizable set as \( t_{-j,m} \), and the last is because \( \pi \) is valid for \( t_{j,m} \). Finally, since \( a_j \in BR_j \left( \text{marg}_{\Theta \times A_{-j}} \pi \right) \) and \( \bar{\pi} \) and \( \pi \) have the same marginal distribution on \( \Theta \times A_{-j} \), it follows that \( a_j \in BR_j (\bar{\pi}) \). Hence, \( \overline{S}_j [\cdot] \) satisfies the best-reply property on \( (\Theta \times \tilde{T}_m^j, \overline{\kappa}^m) \). Thus, by (Dekel et al., 2007, Proposition 4), \( \overline{S}_j [\bar{t}_{j,m}] \subseteq S_j^\infty [\bar{t}_{j,m}] \), and moreover, since \( \overline{S}_j [\bar{t}_{j,m}] = S_j^\infty [t_{j,m}] \), we obtain \( S_j^\infty [t_{j,m}] \subseteq S_j^\infty [\bar{t}_{j,m}] \) and because every \( t_j \in \bar{t}_{j,m} \) has the same rationalizable set as \( t_{j,m} \), we get \( S_j^\infty [t_j] \subseteq S_j^\infty [f_{j,m} (t_j)] \).

**Step 3.** \( \lim_{m \to \infty} \sup_{t_j \in T_j^*} d_j (f_{j,m} (t_j), t_j) = 0 \).

Let \( t_j \in T_j^* \) and \( f_{j,m} (t_j) = \bar{t}_{j,m} \). We show that the \( k^{th} \)-order belief of \( \bar{t}_{j,m} \) (viewed as a type in the model \( (\Theta \times \tilde{T}_m^j, \overline{\kappa}^m) \)) converges to \( t_j^k \) and the convergence is uniform in \( t_j \). We
prove this by induction on $k$. For $k = 1$, observe that by (2) $\tilde{t}_{j,m}^k = t_{j,m}^k$. By (1), $d_j(t_j, \tilde{t}_{j,m}) = d_j(t_1, t_{j,m}) < 1/m$, and since $d_j^1(\tilde{t}_{j,m}^1, t_j^1) \leq d_j(t_{j,m}, t_j)$, $\lim_{m \to \infty} \sup_{t_j \in T_j^*} d_j^1(\tilde{t}_{j,m}^1, t_j^1) = 0$.

Now consider $k > 1$. Let $\varepsilon \in (0, 1)$ and we show that for sufficiently large $m$, $d_j^k(\tilde{t}_{j,m}^k, t_j^k) < \varepsilon$ for all $t_j \in T_j^*$. By the induction hypothesis, there is some $m(\varepsilon)$ such that for any $m > m(\varepsilon)$, $\max_{j' \neq j} d_j^{k-1}(f_j'(t_{j'}^k), t_{j'}^{k-1}) < \varepsilon/2$ for all $t_{j'} = (t_{j'}^k)_{j' \neq j} \in T_{j'}^*$. Consider $m > \{2/\varepsilon, m(\varepsilon)\}$. Recall that $d_j^k$ is the Prohorov metric on the space of all $k$th-order beliefs. Since $\tilde{t}_{j,m}$ is a finite type, it suffices to verify that for each $(\theta, \tilde{t}_{j,m}^{k-1})$ on the support of $\tilde{t}_{j,m}$, we have

$$\tilde{t}_{j,m}^k[(\theta, \tilde{t}_{j,m}^{k-1})] = \kappa_{\tilde{t}_{j,m}}^k\left(\{ (\theta, t_{j,m}) : f_{j,m}(t_{j,m}) \tilde{t}_{j,m}^{k-1} = \tilde{t}_{j,m}^{k-1} \}\right)$$

$$\leq \kappa_{\tilde{t}_{j,m}}^k(\theta, \tilde{t}_{j,m}^{k-1})^{\varepsilon/2}$$

$$< d_j^k((\theta, \tilde{t}_{j,m}^{k-1}), \varepsilon) + \varepsilon$$

where the first equality follows from (2); the first inequality follows because $f_{j,m}(t_{j,m}) \tilde{t}_{j,m}^{k-1} = \tilde{t}_{j,m}^{k-1}$ implies $\max_{j' \neq j} d_j^{k-1}(f_j'(t_{j',m}), t_{j'}^{k-1}) < \varepsilon/2$ (since $m > m(\varepsilon)$); the second follows because by (1), $d_j^k(\tilde{t}_{j,m}^k, t_j^k) < 1/m < \varepsilon/2$ (since $m > 2/\varepsilon$). Thus, for $m > \{2/\varepsilon, m(\varepsilon)\}$, $d_j^k(f_{j,m}(t_j)^k, \tilde{t}_{j,m}^k) < \varepsilon$ for all $t_j \in T_j^*$. Since $\varepsilon > 0$ is arbitrary, the induction step follows.

### A.3 Proof of Theorem 1

**Theorem 1** RURA holds iff for any type $t_i \in T_i^*$ and action $a_i \in S_i^\infty[t_i]$, there is a sequence of types $(t_i(m))_{m=1}^\infty$ such that $\lim_{m \to \infty} t_i(m) = t_i$ and $S_i^\infty[t_i(m)] = \{a_i\}$, $\forall m$.

**Proof.** The "if" part is immediate. We now prove the "only if" part. Suppose RURA holds. We first prove the result for any finite type. Let $t_i$ be a finite type and $a_i \in S_i^\infty[t_i]$. By Lemma 1, there is a sequence of types $(t_i(m))_{m=1}^\infty$ such that $\lim_{m \to \infty} t_i(m) = t_i$ and $a_i \in W_i^\infty[t_i(m)]$. Moreover, for each $m$, by Lemma 2, there is a sequence of types $(t_i(m,k))_{k=1}^\infty$ such that $\lim_{k \to \infty} t_i(m,k) = t_i(m)$ and $\{a_i\} = S_i^\infty[t_i(m,k)]$. Since $T_i^*$ is a metric space, there is some sequence of types $(t_i(m,k_m))_{m=1}^\infty$ such that $\lim_{m \to \infty} t_i(m,k_m) = t_i$ and $S_i^\infty[t_i(m,k_m)] = \{a_i\}$.

Now consider an infinite type $t_i \in T_i^*$ and an action $a_i \in S_i^\infty[t_i]$. By Lemma 3, there is a sequence of finite type $t_i(m)$ such that $a_i \in S_i^\infty[t_i(m)]$ for every $m$ and $\lim_{m \to \infty} t_i(m) = t_i$. 

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For each $m$, since $t_i(m)$ is a finite type, by our previous step there is a sequence of types $(t_i(m,k))_{k=1}^\infty$ such that $\lim_{k \to \infty} t_i(m,k) = t_i(m)$ and $\{a_i\} = S_i^{\infty}[t_i(m,k)]$. Since $T_i^r$ is a metric space, there is some sequence of types $(t_i(m,k))_{m=1}^\infty$ such that $\lim_{m \to \infty} t_i(m,k) = t_i$ and $S_i^{\infty}[t_i(m,k_m)] = \{a_i\}$. ■

A.4 Proof of Lemma 4

**Lemma 4** There are some set of payoff parameters $\Theta$ and utility functions $(U_i)_{i \in N}$ with $U_i : \Theta \times Z \to \mathbb{R}$ such that $(\Gamma, (U_i)_{i \in N})$ satisfies EF-richness.

**Proof.** Let $\Theta = \{\theta_{a_i} : a_i \in A_i, i \in N\}$. For any history $h \in H_i$, we define

$$\alpha(h, a_i) = |\{h' \in H_i : h' \in H(a_i) \text{ and } h' \text{ is a subhistory of } h\}|.$$

where $|\cdot|$ denotes the cardinality of "\(\cdot\)". Define

$$U_i(\theta_{a_i}, z) \equiv \alpha(z, a_i), \forall z \in Z$$

and for any $j \neq i$ and $a_i \in A_i$, define $U_j(\theta_{a_i}, \cdot)$ arbitrarily. We now show that $(\Gamma, (U_i)_{i \in N})$ satisfies EF-richness.

Let $a_i, a'_i \in A_i$ and $a_{-i} \in A_{-i}$. Suppose that $a'_i(h) \neq a_i(h)$ for some subhistory $h$ of $z(a_i,a_{-i})$ and we will show $\alpha(z(a_i,a_{-i}), a_i) \geq \alpha(z(a'_i,a_{-i}), a_i) + 1$. Since the game is finite, we may assume without loss of generality that $h$ is the shortest among the subhistories of $z(a_i,a_{-i})$ at which $a_i$ and $a'_i$ differ.

First, since $a'_i(h) \neq a_i(h)$, for any subhistory $h'$ of $z(a'_i,a_{-i})$ such that $(h,a'_i(h))$ is a further subhistory $h'$, $h' \notin H(a_i)$ and hence

$$\alpha(z(a'_i,a_{-i}), a_i) = \alpha(h,a_i). \quad (4)$$

Second, since $(h,a_i(h)) \in H(a_i)$ is a subhistory of $z(a_i,a_{-i})$ but is not a subhistory of $h$,

$$\alpha(z(a_i,a_{-i}), a_i) \geq \alpha(h,a_i) + 1. \quad (5)$$

By (4)–(5), we conclude that $\alpha(z(a_i,a_{-i}), a_i) \geq \alpha(z(a'_i,a_{-i}), a_i) + 1$. ■
A.5 Proof of Theorem 3

Theorem 3 If the extensive-form game \((\Gamma, (U_i)_{i \in N})\) satisfies EF-richness, then \(ISR_i^\infty [t_i] = S_i^\infty [t_i]\) for all \(t_i \in T_i^*\).

Proof. Clearly, if \(a_i\) is a sequential best reply to a conditional probability system \(\pi\) on \(\Theta \times T_i^* \times A_{-i}\), then \(\text{marg}_{\Theta \times T_i^*} \pi (\cdot | \phi) = \kappa_{t_i} a_i \in BR_i (\text{marg}_{\Theta \times A_{-i}} \pi (\cdot | \phi))\). Hence, \(ISR_i^k [t_i] \subseteq S_i^k [t_i]\) for every \(k \geq 0\) and therefore \(ISR_i^\infty [t_i] \subseteq S_i^\infty [t_i]\) for every \(t_i \in T_i^*\). Conversely, suppose that \(t_i \in T_i^*\) and \(a_i \in S_i^\infty [t_i]\). We show \(a_i \in ISR_i^\infty [t_i]\). By Theorem 2, there is a sequence of types \((t_i (m))_{m=1}^\infty\) such that \(\lim_{m \to \infty} t_i (m) = t_i\) and \(\{a_i\} = S_i^\infty [t_i (m)]\) for every \(m\). Since \(\emptyset \neq ISR_i^\infty [t_i] \subseteq S_i^\infty [t_i]\) for every \(t_i \in T_i^*\), \(\{a_i\} = ISR_i^\infty [t_i (m)]\) for every \(m\). Since \(ISR_i^\infty [\cdot]\) is upper hemicontinuous on \(T_i^*\), we have \(a_i \in ISR_i^\infty [t_i]\). Thus, \(ISR_i^\infty [t_i] = S_i^\infty [t_i]\) for all \(t_i \in T_i^*\). \(\blacksquare\)

References


